Robustness of the Independent Modal-Space Control Method

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The effect of parameter uncertainties on the control system performance of distributed-parameter systems is examined. Because in general, the parameters contained in the equations of motion of the actual distributed system are, not known accurately, control forces designed on the basis of a postulated model will not control the actual distributed system effectively. In this paper it is shown by means of a stability theorem that, when the independent modal-space control (IMSC) method is used in conjunction with modal filters, any errors in the system parameters cannot lead to instability of the closed-loop system, so that the control system is very robust. A perturbation analysis is proposed for the computation of the closed-loop poles of large-order systems in the presence of parameter changes.

I. Introduction

THE motion of a distributed-parameter system is governed generally by a set of simultaneous partial differential equations of motion.[1] The parameters contained in the equations of motion are, in general, continuous functions of the spatial variables. For flexible structures, these parameters represent mass, stiffness, and damping distributions.

To control the distributed system, one must construct a mathematical model of the distributed system. The control forces then are designed on the basis of the mathematical model. A common approach to modeling is to convert the partial differential equations of the distributed system into an infinite set of ordinary differential equations.[1-3] Then, a limited number of modes (generally the lowest) are retained for control.

In designing the control system, one assumes that the eigensolution associated with the controlled modes is known with sufficient accuracy, which, in turn, assumes that the system parameters are known accurately. Errors in the eigensolution produce errors in the design and implementation of the controls. Hence, the question arises whether the control system designed on the basis of system parameters that are in error can control the actual system effectively; i.e., whether the control system is robust. The answer clearly depends on the degree of inaccuracy in the estimated state of the distributed system. For cases when this error is not very large, one intuitively expects very small deviations from the control system performance. In general, one should make some allowance in the control system design for parameter uncertainties. For cases when the parameters contained in the equations of motion, such as the mass and stiffness distributions, are known to within a multiplicative constant, only the system eigenvalues change and the eigenfunctions retain the same shape.[4] A sensitivity study, based on a perturbation analysis treating the parameter errors as perturbations reveals that if the independent modal-space control (IMSC) method is used in conjunction with modal filters, the control system is relatively insensitive to parameter errors.[4] When the spatial distributions of the system parameters are not known, however, both the estimated eigenvalues and eigenfunctions tend to differ from their actual values, so that the sensitivity analysis of Ref. 4 is not applicable.

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It should be noted that the manner in which the parameter errors affect the system performance depends on the way the modal quantities are extracted from the system output and the way in which the actual controls are implemented. There are two methods that permit extraction of the modal quantities from sensors data, observers,[5] and modal filters.[4] In this paper, modal filters are used. Because the modal filters eliminate observation spillover and they are implemented by performing weighted integrations along the distributed domain, when modal filters are used any error associated with parameter uncertainties tends to be minimized.

One can distinguish between two types of modal control methods, namely, coupled controls[6] and IMSC.[4,7-9] When the IMSC method is used, the control forces are designed in the modal space, instead of the actual space. The eigenfunctions are used only when the actual control forces are synthesized from the modal control forces, so that the interaction between the actual control forces and modal coordinates is minimized.

In this paper, the sensitivity of the IMSC method in conjunction with modal filters to errors in the system parameters is examined. The errors arise from uncertainties in the mass and stiffness distributions and their effect is investigated by means of a stability theorem, a series of eigenvalue analyses, and comparisons of the system response. It is shown that the control system is relatively robust, and its stability is not affected adversely by parameter errors. It is also shown that when errors in the mass and stiffness distributions are small these errors can be considered as perturbations on the original system, so that the sensitivity of the actual closed-loop system can be studied by means of a perturbation analysis.

II. Equations of Motion for Distributed-Parameter Systems

The equation of motion for a distributed-parameter system (DPS) can be written as[1]

$$Lu(P,t) + M(P)\partial^2 u(P,t)/\partial t^2 = f(P,t) \tag{1}$$

which must be satisfied at every point P of the domain D, where $u(P,t)$ is the displacement of an arbitrary point P, L is a linear differential self-adjoint positive definite operator of order $2p$, $M(P)$ is the distributed mass, and $f(P,t)$ is the distributed control. The displacement $u(P,t)$ is subject to the boundary conditions $B_i u(P,t) = 0$ $(i = 1,2,...,p)$, where B_i $(i=1,2,...,p)$ are linear differential operators. The solution of the associated eigenvalue problem consists of a denumerably infinite set of eigenvalues Λ_r and associated eigenfunctions ϕ_r $(r=1,2,...)$. The eigenvalues are related to the natural frequencies ω_r of the undamped oscillation by

$\Lambda_r = \omega_r^2$ $(r=1,2,...)$. The eigenfunctions can be normalized so as to satisfy $\int_D M\phi_r\phi_s dD = \delta_{rs}$, $\int_D \phi_s L \phi_r dD = \Lambda_r \delta_{rs}$ $(r,s = 1,2,...)$, where δ_{rs} is the Kronecker delta.

Using the expansion theorem[1]

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P) u_r(t) \tag{2}$$

where $u_r(t)$ are *modal amplitudes*, Eq. (1) can be replaced by the infinite set of ordinary differential equations

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t), \quad r=1,2,... \tag{3a}$$

known as *modal equations*, where

$$f_r(t) = \int_D \phi_r(P) f(P,t) dD, \quad r=1,2,... \tag{3b}$$

are *modal control forces*.

Implicit in the preceding developments is the assumption that the full infinite set of eigenfunctions is available readily. This is true only in the case in which the eigenvalue problem admits a closed-form solution. In most practical cases closed-form solutions are not possible, so that one must be content with an approximate solution, which invariably requires the numerical solution of an algebraic eigenvalue problem replacing the differential eigenvalue problem. However, algebraic eigenvalue problems are defined over a finite-dimensional vector space, so that only a finite number of lower eigenvalues and eigenfunctions can be computed. Of the computed eigenvalues and eigenfunctions, at times less than one half are accurate, where the accurate ones correspond to the lower eigenvalues.[1-3] Because it is seldom feasible to control the entire infinity of modes, we must truncate series (2), which amounts to replacing a distributed-parameter model by a discrete one, a process known as discretization. Common sense dictates that only the accurate modes be included in the discretized model. We shall refer to the finite number of modes included in the discretized model as *modeled modes,* which implies that there is an infinity of *unmodeled modes.*

III. Control System Design

We assume that all the modeled modes are controlled, so that the system modes can be classified as *controlled* (modeled) and *uncontrolled,* or *residual* (unmodeled). It will be shown later that if modal filters are used, all the system modes can actually be treated as modeled.[4] It should be noted that the contribution of the lower modes to the overall motion of a DPS is generally much larger than the contribution of the higher modes, so that one is further justified in controlling the lower modes. We propose to control the lowest n modes.

Equations (3a) have the appearance of an infinite set of independent second-order differential equations, and in the absence of feedback control forces they indeed are. This decoupling is referred to as internal. If the feedback forces $f_r(t)$ depend on all the modal coordinates

$$f_r = f_r(u_1, \dot{u}_1, u_2, \dot{u}_2, ...), \quad r=1,2,...,n \tag{4}$$

where n denotes the number of controlled modes, then Eqs. (3a) are *externally coupled.* This is the case of *coupled controls.* In the special case in which f_r depends on u_r and $\dot{u}_r$ alone

$$f_r = f_r(u_r, \dot{u}_r), \quad r=1,2,...,n \tag{5}$$

Equations (3a) become *internally and externally decoupled.* This is the essence of *independent modal-space control* (IMSC). A comparison of IMSC and coupled modal control methods indicates that IMSC is more desirable from both design and computational viewpoints.[10]

The IMSC method permits the design of the control system for each mode separately, where the design takes place in the modal space. This is because in the IMSC method the modal control forces are computed first. Then, the actual control forces are obtained from the modal forces by a linear transformation, as will be shown shortly.

In the case of distributed controls, one can eliminate control spillover into the uncontrolled modes entirely. When distributed controls are not realizable, we can carry out the control task by means of discrete actuators. Discrete actuator forces can be treated as distributed by writing

$$f(P,t) = \sum_{j=1}^{m} F_j(t)\delta(P-P_j) \tag{6}$$

where $F_j(t)$ are the actuator forces and $\delta(P-P_j)$ are spatial Dirac delta functions $(j=1,2,...,m)$, where m is the number of actuators. Introducing Eq. (6) into Eqs. (3b), we obtain

$$f_r(t) = \sum_{j=1}^{m} \phi_r(P_j) F_j(t), \quad r=1,2,... \tag{7}$$

Equations (7) imply that the actual control forces $F_j(t)$ $(j=1,2,...,m)$ produce modal forces for every single mode. Of course, these forces are selected such that the vibratory motion of the controlled modes is suppressed. In the process, because $f_r(t) \neq 0$ $(r=n+1,n+2,...)$, one excites the uncontrolled modes also, a phenomenon that has come to be known as *control spillover.*

Next, let us introduce the vectors

$$u_C(t) = [u_1(t)\ u_2(t) ... u_n(t)]^T$$

$$f_C(t) = [f_1(t)\ f_2(t) ... f_n(t)]^T$$

$$F(t) = [F_1(t)\ F_2(t) ... F_m(t)]^T \tag{8}$$

and the matrices

$$\Lambda_C = \mathrm{diag}(\omega_1^2\ \omega_2^2 ... \omega_n^2)$$

$$B = [\phi_i(P_j)], \quad i=1,2,...,n; \quad j=1,2,...,m \tag{9}$$

Then, Eqs. (3a) for the controlled modes alone can be expressed as

$$\ddot{u}_C(t) + \Lambda_C u_C(t) = f_C(t) \tag{10}$$

where, from Eqs. (7),

$$f_C(t) = BF(t) \tag{11}$$

The actual control forces $F(t)$ can be synthesized from the modal control forces $f_C(t)$ by inverting Eq. (11), or

$$F(t) = B^{-1} f_C(t) \tag{12}$$

Equation (12) requires that B be a square nonsingular matrix, which implies that $m=n$, or the number of actuators must be equal to the number of controlled modes.

For feedback controls, one must extract the modal displacements and velocities associated with the controlled modes from the system output. To this end, we can make use of the second part of the expansion theorem[1] and write

$$u_r(t) = \int_D M(P)\phi_r(P) u(P,t) dD$$

$$\dot{u}_r(t) = \int_D M(P)\phi_r(P) \dot{u}(P,t) dD, \quad r=1,2,... \tag{13}$$

Equations (13) can be regarded as *modal filters.* They permit the extraction of the modal displacement $u_r(t)$ and modal velocity $\dot{u}_r(t)$ from measurements of the actual displacement $u(P,t)$ and actual velocity $\dot{u}(P,t)$ at every point P of the domain D and at all times t. Hence, if modal filters are used, then all the system modes are observable and can be regarded as modeled. Having $u_r(t)$ and $\dot{u}_r(t)$, one can generate the modal controls $f_r(t)$ $(r=1,2,...)$.

The modal filters described by Eqs. (13) require distributed measurements of $u(P,t)$ and $\dot{u}(P,t)$ and integration over the spatial domain. It is shown in Ref. 4 that it is possible to implement modal filters by using a finite number of discrete sensors only and by spatially interpolating the sensors output to obtain displacement and velocity profiles. The interpolation functions can be chosen from the finite-element method and the integrations can be performed as off-line computations.[11] These displacement and velocity profiles thus generated can then be used to extract the modal displacements and velocities associated with the controlled modes. Hence, a finite number of sensors can estimate a given number of modal displacements and velocities accurately and, furthermore, it is relatively easy to determine this number. The estimated modal coordinates become[11]

$$\hat{u}_r(t)=\sum_{i=1}^{k} I_{ir}^T y_i(t),\quad \hat{\dot{u}}_r(t)=\sum_{r=1}^{k} I_{ir}^T \dot{y}_i(t),\qquad r=1,2,...,n \tag{14}$$

where $\hat{u}_r$ and $\hat{\dot{u}}_r$ are the estimated modal displacements and velocities, respectively, k is the number of subdomains, and $y_i(t)$ is a vector of measurements at the boundaries of the ith subdomain. In addition,

$$I_{ir}=\int_{D_i} M(P)\phi_r(P)L\,\mathrm{d}D,\quad k=1,2,...,k;\quad r=1,2,...,n \tag{15}$$

where L is a vector of interpolation functions from the finite-element method. It is clear from Eqs. (15) that the integrations can be performed as off-line computations, so that modal filters can be implemented with relative ease.

IV. Control of the Actual System

The preceding analysis is based on the assumption that the parameters contained in the equations of motion are known accurately. Quite often this is not the case. In practice, the nature of and the parameters contained in the mass and stiffness distributions are known only approximately. In this section, we wish to examine the effects of applying controls on a distributed system whose parameters are not known accurately.

We work on the basis of the mass and stiffness operators $M(P)$ and L, when, in fact, the actual mass and stiffness operators are $M^*(P)$ and L^*. We refer to these two systems as *postulated system* and *actual system*, respectively. It is clear that the eigensolution associated with the postulated system is different from the eigensolution associated with the actual system. Clearly, the eigenfunctions $\phi_r (r=1,2,...)$ belonging to the postulated system do not represent eigenfunctions of the actual DPS, but only comparison functions.[1]

We wish to express the motion of the actual system in terms of generalized coordinates $u_1,u_2,...$ belonging to the postulated system. To this end, we reconsider the expansion theorem, Eq. (2), but, unlike in Eq. (2), $\phi_r(P)$ $(r=1,2,...)$ now represent comparison functions for the actual system. The equation of motion for the actual DPS is given by

$$L^*u(P,t)+M^*(P)\frac{\partial^2 u(P,t)}{\partial t^2}=f(P,t) \tag{16}$$

where L^* is self-adjoint. Introducing Eq. (2) into Eq. (16), multiplying by $\phi_s(P)$ and integrating over the domain D, we obtain the differential equations of motion of the actual system

$$M^*\ddot{u}(t)+K^*u(t)=f(t) \tag{17}$$

where matrices M^* and K^* have the entries

$$M^*_{rs}=M^*_{sr}=\int_D M^*(P)\phi_r(P)\phi_s(P)\,\mathrm{d}D\quad r,s=1,2,... \tag{18a}$$

$$K^*_{rs}=K^*_{sr}=\int_D \phi_r(P)L^*\phi_s(P)\,\mathrm{d}D,\quad r,s=1,2,... \tag{18b}$$

and

$$u(t)=[u_1\,u_2...]^T \tag{18c}$$

$$f(t)=[f_1\,f_2...]^T \tag{18d}$$

in which we note that the matrices M^* and K^* are, in general, not diagonal. However, when $M^*(P)=M(P)$ and $L^*=L$, the matrices M^* and K^* do become diagonal and Eq. (17) reduces to the vector form of Eqs. (3a). The eigenvalues of the open-loop actual system can be obtained by solving the eigenvalue problem

$$(\lambda^2 M^*+K^*)u=0 \tag{19}$$

In reality, M^* and K^* are infinite-dimensional matrices, but they must be truncated for computational purposes.

Let us assume that we can ignore the higher modes and consider a truncated model corresponding to the finite-dimensional set of controlled modes. Quite often, for an accurate description and effective control of the distributed system, the order of the truncated model must be very large, so that the effect of the higher modes on the overall response becomes insignificant.

To examine the behavior of the actual system qualitatively in the case of IMSC, let us consider linear proportional control of the form

$$f_C(t)=G_1u_C(t)+G_2\dot{u}_C(t) \tag{20}$$

where G_1 and G_2 are diagonal gain matrices. Introducing Eq. (20) into Eq. (17) and considering an n-degree-of-freedom truncated system, we obtain the closed-loop equations

$$M^*\ddot{u}_C(t)-G_2\dot{u}_C(t)+(K^*-G_1)u_C(t)=0 \tag{21}$$

where M^* and K^* are now of dimensions $n\times n$.

In writing the closed-loop equations in the form of Eq. (21), it is implied that one can extract $u_C(t)$ and $\dot{u}_C(t)$ from the system output with no observation spillover. This is possible only when modal filters are used. Indeed, if modal filters were not available, one would have to use an observer to estimate $u_C(t)$ and $\dot{u}_C(t)$. In this case, Eqs. (20) and (21) would be different.

For reasons that will become evident shortly, let us now examine the sign properties of G_1 and G_2. The matrix G_2 is negative definite and symmetric. This is so, because the closed-loop poles of the controlled modes must lie in the left side of the s plane. The sign properties of G_1 vary from one control method to another. For velocity feedback only, $G_1=0$. In the case of optimal control, G_1 is negative definite. The modal control forces for optimal control are given by[4]

$$f_r(t)=\omega_r(\omega_r-\sqrt{\omega_r^2+R_r^{-1}})u_r(t)-[2\omega_r(-\omega_r+\sqrt{\omega_r^2+R_r^{-1}})+R_r^{-1}]^{1/2}\dot{u}_r(t),\quad r=1,2,...,n \tag{22}$$

where R_r is a positive number to be chosen by the analyst. The displacement feedback term can be expressed as $G_{1rr} = \omega_r^2(1 - \sqrt{1 + R_r^{-1}/\omega_r^2})$, which clearly shows that $G_{1rr} < 0$. In the case of the pole-allocation method, the sign definiteness of G_1 depends on the desired closed-loop poles, so that one can select the poles such that G_1 is negative definite.

The matrices M^* and K^* are symmetric and positive definite, because $M^*(P)$ is a positive function and L^* is a self-adjoint, positive definite operator. The sign defînteness of M^* and K^* is not affected by truncation. Because G_1 is negative definite and diagonal, the matrix $K^* - G_1$ is symmetric and positive definite. At this point, we wish to invoke the Kelvin-Tait-Chetaev stability theorem,[12,13] which, for our purposes, can be stated as follows: Given the matrix equation

$$A\ddot{u}(t) + D\dot{u}(t) + Ku(t) = 0 \tag{23}$$

If A and D and K are symmetric positive definite matrices, then the trivial solution is asymptotically stable. In view of this theorem, we conclude that *the closed-loop system described by Eq. (21) is guaranteed to be stable.* Hence, based on the analysis of a truncated system, we can state the following. *Theorem: any errors in the mass and stiffness distributions cannot lead to instabilities in the closed-loop system, provided G_1 is chosen as a negative definite matrix.* This is a very important result, because *it proves the robustness of the control system designed by the IMSC method.* This proof assumes that modal filters are used to extract the modal quantities from the system output, so that no observation spillover exists.

It should be noted that the preceding theorem is applicable only when IMSC is used. This is true because in the case of coupled controls G_1 is not symmetric so that the Kelvin-Tait-Chetaev theorem is not applicable. Hence, it can be said that the use of IMSC in conjunction with modal filters has the desirable effect of minimizing parameter uncertainties.

Note that the aforementioned results are not applicable to cases in which the residual modes are included in the system equations. Indeed, considering Eqs. (8-12) and (20), we can write

$$f(t) = G_1^* u(t) + G_2^* \dot{u}(t) \tag{24}$$

where

$$G_1^* = \left[\begin{array}{c|c} G_1 & 0 \\ \hline B_R B^{-1} G_1 & 0 \end{array}\right] \tag{25a}$$

$$G_2^* = \left[\begin{array}{c|c} G_2 & 0 \\ \hline B_R B^{-1} G_2 & 0 \end{array}\right] \tag{25b}$$

and

$$B_{Rij} = \phi_i(P_j), \qquad i = n+1, n+2, \ldots; \qquad j = 1, 2, \ldots, n \tag{25c}$$

The closed-loop equations for the actual DPS become

$$M^* \ddot{u}(t) - G_2^* \dot{u}(t) + (K^* - G_1^*) u(t) = 0 \tag{26}$$

If one truncates Eq. (26) and retains some of the residual modes in the truncated system, G_1^* and G_2^* are no longer symmetric, so that the stability theorem mentioned above does not apply. To control a DPS effectively, however, one must control a large number of modes. The number of controlled modes can be chosen such that the contribution of the residual modes to the overall motion becomes insignificant. Hence, if an adequate number of modes is controlled, then the residual modes can be ignored and one can safely make use of the stability theorem stated above.

V. Sensitivity of the Closed-Loop System

It was demonstrated in Sec. IV that errors in the mass and stiffness distributions cannot destabilize the closed-loop system, although such errors are likely to degrade the system performance. In this section, we wish to examine the sensitivity of the control system to parameter errors quantitatively.

One method of examining the effect of errors in parameters on the closed-loop system is to compare the closed-loop poles of the postulated and actual systems. If the IMSC method is used, then the closed-loop poles of the postulated system can be calculated independently for each mode. To obtain the closed-loop poles of the actual system, however, one must solve the eigenvalue problem associated with Eq. (26), namely,

$$(\lambda^2 M^* + \lambda G_1^* + K^* - G_2^*) x = 0 \tag{27}$$

where λ are the eigenvalues and x are the eigenvectors. As stated earlier, the matrices M^*, K^*, G_1^*, and G_2^* must be truncated before an eigenvalue analysis can be carried out. It is intuitively clear that, as the order of the eigenvalue problem is increased, one can assess the effect of errors in the system parameters more accurately.

Let us assume that the order of the eigenvalue problem (27) is of order k, $k > n$. Because λ appears in the eigenvalue problem (27) to the second power, it is convenient to reduce it to an eigenvalue problem in which λ appears only linearly. This latter eigenvalue problem is of order $2k$ and is defined by two real general matrices. It is obtained by casting the eigenvalue problem in state form. Note that the reason for reducing the eigenvalue problem to one in which λ appears linearly is that the most efficient computational algorithms for the eigensolution are in this form. When $2k$ is large, the solution of an eigenvalue problem defined by real general matrices is, in general, an unpleasant task from a computational point of view. Because in our case the eigensolution associated with the postulated system is known, we can resort to a perturbation technique to obtain the eigensolution of the actual system with relative ease, especially when the errors in the system parameters are relatively small.

In view of the foregoing let us rewrite Eq. (26) in the form

$$(I + \Delta M)\ddot{u}(t) + G_2^* \dot{u}(t) + (\Lambda + \Delta K - G_1^*) u(t) = 0 \tag{28}$$

in which we have substituted the relations

$$M^* = I + \Delta M, \qquad K^* = \Lambda + \Delta K \tag{29}$$

where I is an identity matrix, ΔM is a real symmetric matrix, Λ is a diagonal matrix containing the eigenvalues of the postulated system, and ΔK is a real symmetric matrix. Note that ΔM and ΔK are increments in the mass and stiffness matrices representing the difference between the actual system and the postulated system. When errors in the mass and stiffness distributions are small, we can assume that the entries of ΔM and ΔK are of one order of magnitude smaller than the entries of I and Λ, respectively.

Introducing the auxiliary variable vector $v(t) = \dot{u}(t)$, we obtain

$$(I + \Delta M)\dot{v}(t) - G_2^* v(t) + (\Lambda + \Delta K - G_1^*) u(t) = 0 \tag{30}$$

Because ΔM is small, we can write the truncated binomial expansion

$$(I + \Delta M)^{-1} \cong I - \Delta M \tag{31}$$

Premultiplying Eq. (30) by $(I+\Delta M)^{-1}$ and making use of Eq. (31), we obtain the system dynamics in the state form

$$\begin{bmatrix} \dot{v} \\ \hline \dot{u} \end{bmatrix} = A \begin{bmatrix} v \\ \hline u \end{bmatrix} \tag{32}$$

where

$$A = \left[\begin{array}{c|c} (I-\Delta M)G_2^* & -(I-\Delta M)(\Lambda+\Delta K-G_1^*) \\ \hline I & 0 \end{array}\right] \tag{33}$$

The coefficient matrix A can be written in the form

$$A \cong A_0 + A_1 \tag{34}$$

where

$$A_0 = \left[\begin{array}{c|c} G_2^* & -\Lambda+G_1^* \\ \hline I & 0 \end{array}\right] \tag{35a}$$

$$A_1 = \left[\begin{array}{c|c} -\Delta MG_2^* & -\Delta K+\Delta M(\Lambda-G_1^*) \\ \hline 0 & 0 \end{array}\right] \tag{35b}$$

and we note that the entries of A_1 are of one order of magnitude smaller than the entries of A_0. The eigenvalue problem associated with Eq. (32) can be written as

$$\lambda z = Az \cong (A_0 + A_1)z \tag{36}$$

and its solution can be obtained with relative ease by using a given perturbation technique,[14] whereby the eigensolution of A can be computed on the basis of the eigensolution of A_0, without solving a new eigenvalue problem. Note that the eigensolution of A_0 is available readily. Indeed, when the truncated system consists of the controlled modes only, the eigenvalues of A_0 are the closed-loop poles of the postulated system and the right eigenvectors of A_0 are

$$z_j = [0\ 0 \ldots 1\ 0 \ldots 0 \,|\, 0\ 0 \ldots p_j\ 0 \ldots 0]^T, \quad j=1,2,\ldots,2k \tag{37}$$

where unity is in the jth position and the closed-loop pole p_j is in the $(k+j)$th position. When some of the higher modes are retained in the truncated system, the eigenvalues of A_0 are

$$\lambda_j = p_j,\ j=1,2,\ldots,n;\ \lambda_j = i\omega_j,\ j=n+1,n+2,\ldots,k \tag{38}$$

where $i^2=-1$. On the other hand, the eigenvectors of A_0 no longer have closed-form expressions. However, if the eigenvalues of a matrix are known, then the eigenvectors can be obtained without much computational effort.[15] It should be noted that the perturbation analysis is feasible only when IMSC is used in conjunction with modal filters. Indeed, because in the case of coupled controls G_1 and G_2 are fully populated matrices, one cannot obtain the eigensolution of A_0 in closed-form, so that one must use a computational algorithm for the eigensolution of A_0. This, of course, defeats the whole idea behind a perturbation analysis. Indeed, the advantage of the perturbation method is that one can design a control system for the postulated values of the mass and stiffness distributions, and then alter these distributions to compare the performance of the closed-loop system without solving a new eigenvalue problem.

Another method of observing the effect of errors in the system parameters on the performance of the actual system is to compare the response of the postulated and actual DPS's for the same control forces. This approach will be demonstrated via a numerical example in the next section.

VI. Numerical Example

As a numerical example, let us consider the control of the bending vibration of a tapered cantilever beam. We assume that the postulated system has the parameters

$$M(x)=1-0.2x/\ell, \quad EI(x)=1-0.2x/\ell, \quad \ell=5 \tag{39}$$

where x is the spatial variable and ℓ the beam length. The stiffness operator is $L=\mathrm{d}^2[EI(x)\mathrm{d}^2/\mathrm{d}x^2]/\mathrm{d}x^2$. The eigensolution of the postulated system was computed by means of a finite-element analysis yielding a system of order 100. The lowest 20 eigenvalues and corresponding eigenfunctions were considered as accurate, so that the control system design is based on 20 modes.

The control forces were designed based on the IMSC method. The optimal control gain parameters were taken as[4]

$$Q_r = w_r^2 I, \quad R_r = 20, \quad r=1,2,\ldots,20 \tag{40}$$

The modal control forces are given by Eq. (22).

Let us now consider the actual DPS, where the actual mass and stiffness distributions are

$$M^*(x)=1.1-0.25x/\ell, \quad EI^*(x)=1.2-0.35x/\ell, \quad \ell=5 \tag{41}$$

The 20×20 matrices M^* and K^* were computed using Eqs. (18a) and (18b). Then, the eigensolution of the closed-loop system was obtained by transforming Eq. (27) to state form and using an algorithm for real arbitrary matrices.[15] Whereas for the relatively low-order system of this example we could use existing algorithms to obtain the eigensolution of the closed-loop system, for a higher-order system this algorithm may not be sufficiently reliable. For such cases, it is advised that the perturbation scheme described in Sec. V be used, especially when one wishes to examine the sensitivity of the control system to different types of errors in the system parameters.

Table 1 compares the closed-loop poles of the postulated and actual DPS's. As can be seen, there is little difference in the closed-loop poles, which indicates that the control system is robust when IMSC is used in conjunction with modal filters. From Eqs. (39) and (41) we see that there is about a 10-

Table 1 Real and imaginary parts of the closed-loop poles

Mode No.	Actual system Real	Actual system Imaginary	Postulated system Real	Postulated system Imaginary
1	−0.1367	0.1513	−0.1464	0.1382
2	−0.1460	0.9296	−0.1575	0.9000
3	−0.1461	2.5535	−0.1580	2.4861
4	−0.1461	4.9790	−0.1581	4.8544
5	−0.1461	8.2137	−0.1581	8.0128
6	−0.1461	12.257	−0.1581	11.961
7	−0.1461	17.109	−0.1581	16.698
8	−0.1461	22.770	−0.1581	22.226
9	−0.1461	29.240	−0.1581	28.543
10	−0.1461	36.520	−0.1581	35.651
11	−0.1461	44.609	−0.1481	43.549
12	−0.1461	53.509	−0.1581	52.238
13	−0.1461	63.219	−0.1581	61.719
14	−0.1461	73.741	−0.1581	71.993
15	−0.1461	85.076	−0.1581	83.060
16	−0.1461	97.225	−0.1581	94.922
17	−0.1461	110.19	−0.1581	107.58
18	−0.1461	123.97	−0.1581	121.04
19	−0.1461	138.58	−0.1581	135.30
20	−0.1460	154.14	−0.1581	150.36

15% difference in the postulated and actual mass and stiffness distributions. The average difference in the real parts of the closed-loop poles is only about 8%, which also is an indication of robustness.

Figure 1 compares the response of the tip of the beam for the postulated and actual systems. The response was obtained for a unit impulse applied at $x_0 = 0.63\ell$. It can be shown that the unit impulse has the same effect as initial modal velocities

$$\dot{u}_r(0) = F_0 \phi_r(x_0), \qquad r = 1,2,.. \tag{42}$$

The response of the tip of the beam was obtained by a transition matrix approach under the assumption that a sufficient number of sensors were used to implement the modal filters. The displacements of the tip of the beam was obtained from the response of each mode using the expansion theorem

$$u(5,t) = \sum_{r=1}^{20} \phi_r(5) u_r(t) \tag{43}$$

As can be seen from Fig. 1, there is very little difference between the response of the postulated and actual systems, which is another clear indication of the robustness of the control system.

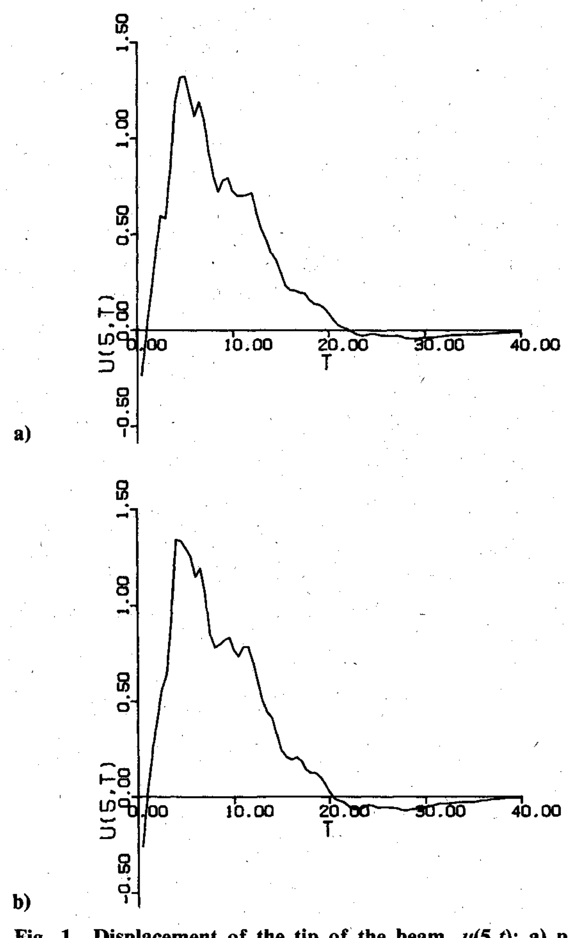

Fig. 1 Displacement of the tip of the beam, $u(5,t)$; a) postulated system and b) actual system.

VII. Conclusions

The effect of parameter uncertainties on the control system performance of distributed-parameter systems is examined. Control forces computed on the basis of a postulated model are applied to the actual distributed-parameter system and the closed-loop behavior is examined qualitatively and quantitatively. It is shown by means of the Kelvin-Tait-Chetaev stability theorem that when the independent modal-space control method is used in conjunction with modal filters, errors in the system parameters cannot lead to instability in the closed-loop systems, so that the control system is relatively insensitive to parameter errors. The eigensolution and the response of the postulated and actual closed-loop systems are compared and it is concluded that the control system is not affected adversely by reasonably small errors in the system parameters.

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References

[1] Meirovitch, L., *Analytical Methods in Vibrations,* The Macmillan Co., New York, N.Y., 1967.

[2] Strang, G. and Fix, G.J., *An Analysis of the Finite Element Method,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973.

[3] Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff-Noordhoff Co., The Netherlands, 1980.

[4] Meirovitch, L. and Baruh, H., "Control of Self-Adjoint Distributed-Parameter Systems," *Proceedings of the AIAA Guidance and Control Conference,* Danvers, Mass., Aug. 1980 pp. 1-10; also, *Journal of Guidance and Control,* Vol. 5, Jan.-Feb. 1982, pp 60-66.

[5] Brogan, W.L., *Modern Control Theory,* QPI Publishers, Inc., New York, N.Y., 1974.

[6] Balas, M.J., "Active Control of Flexible Systems," *Journal of Optimization Theory and Applications,* Vol. 25, No. 3, 1978, pp. 415-436.

[7] Meirovitch, L. and Öz, H., "Modal-Space Control of Distributed Gyroscopic Systems," Paper 78-1421, AIAA/AAS Astrodynamics Conference, Palo Alto, Calif., Aug. 1978; also, *Journal of Guidance and Control,* Vol. 3, 1980, pp. 140-150.

[8] Öz, H. and Meirovitch, L., "Optimal Modal-Space Control of Flexible Gyroscopic Systems," Paper 78-103, *International Astronautical Federation XXIXth Congress,* Dubrovnik, Yugoslavia, Oct. 1978; also, *Journal of Guidance and Control,* Vol. 3, May-June 1980, pp. 218-226.

[9] Meirovitch, L. and Öz, H., "Modal-Space Control of Large Flexible Spacecraft Possessing Ignorable Coordinates," *Journal of Guidance and Control,* Vol. 3, Nov.-Dec. 1980, pp. 569-577.

[10] Meirovitch, L., Baruh, H., and Öz, H., "A Comparison of Control Techniques for Large Flexible Systems," Paper No. 81-195, AAAS/AIAA Astrodynamics Conference, Lake Tahoe, Nev., Aug. 3-5, 1981.

[11] Meirovitch, L. and Baruh, H., "On the Problem of Observation Spillover in Distributed-Parameter Systems," Proceeding of the Third VPI&SU/AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Va., June 1981, Vol. 39, No. 2, Feb. 1982; to appear in the *Journal of Optimization Theory and Applications,* pp. 319-341.

[12] Thomson, W. (Lord Kelvin) and Tait, P.G., *Principles of Mechanics and Dynamics,* Dover Publications, New York, N.Y., 1962, Vol. 1.

[13] Chetaev, N.G., *The Stability of Motion,* Pergamon Press, New York, N.Y., 1961.

[14] Meirovitch, L. and Ryland, G. II, "Response of Slightly Damped Gyroscopic Systems," *Journal of Sound and Vibration,* Vol. 67, No. 1, 1979, pp. 1-19.

[15] Wilkinson, J.H. and Reinsch, C., *Linear Algebra,* Springer-Verlag, New York, N.Y., 1971.